

Discrete Flavor Symmetries and Mass Matrix Textures

M. S. Berger ^{*} and Kim Siyeon [†]

Physics Department, Indiana University, Bloomington, IN 47405, USA

Abstract

We show how introducing discrete Abelian flavor symmetries can produce texture zeros in the fermion mass matrices, while preserving the correct relationships with the low-energy data on quark and lepton masses. We outline a procedure for defining texture zeros as suppressed entries in Yukawa matrices. These texture zeros can account for the coexistence of the observed large mixing in atmospheric neutrino oscillations with a hierarchy in the neutrino masses, and offer the possibility of alignment of the quark and squark mass matrices, and thus giving a solution to the supersymmetric flavor problem. A requirement that the flavor symmetry commutes with the $SU(5)$ grand unified group can be used to explain the lepton mass hierarchies as well as the neutrino parameters, including the large mixing observed in the atmospheric neutrino data. We present one such model that yields a large atmospheric neutrino mixing angle, as well as a solar neutrino mixing angle of order $\lambda \simeq 0.22$.

^{*}Electronic address: berger@gluon.physics.indiana.edu

[†]Electronic address: siyekim@indiana.edu

I. INTRODUCTION

The mechanism that generates the fermion masses is not yet understood. In the standard model (SM) the masses and mixings are simply parameters that can be adjusted to agree with experiment. One hope is that the Yukawa couplings in the SM can be understood more fully when the theory is embedded in a more fundamental theory, and relationships between masses and mixings might then be established. Symmetries based on embedding the gauge symmetries of the SM in larger gauge groups (unified theories) have been used for a long time and some reasonable mass patterns can be derived which are consistent with experiment. These symmetries have come to be called vertical symmetries to distinguish them from the horizontal symmetries (or flavor symmetries) that relate fermions from the different generations. In this paper we show how discrete Abelian horizontal symmetries based on Z_m can account for some of the successful texture patterns.

There are a number of reasons one might want to extend a $U(1)$ flavor symmetry so that it contains an additional discrete Z_m component. (A) The additional discrete symmetry offers a solution to the seemingly inconsistent large mixing observed [1] in the atmospheric neutrino data and a hierarchy in the muon and tau neutrino masses [2]. In models with supersymmetric Abelian flavor symmetries, large $\nu_\mu - \nu_\tau$ mixing is achieved via a light neutrino mass matrix of the form [3]

$$\begin{pmatrix} C & B \\ B & A \end{pmatrix} \frac{v^2}{M},$$

$$A, B, C \sim \mathcal{O}(1), \quad (1)$$

where v is some electroweak scale vacuum expectation value and M is the Majorana neutrino mass scale. The eigenvalues are typically the same order of magnitude. It requires a fine-tuning of the order one parameters A , B and C to achieve large mixing between neutrinos and widely separated neutrino masses. Grossman, Nir and Shadmi [2] advocated using a discrete symmetry to maintain the large mixing angle while achieving very different neutrino masses without fine-tuning. Discrete symmetries had been discussed previously [4], but the more recent experimental results indicating large mixing in the atmospheric (and perhaps solar) neutrino oscillations made this technique especially interesting. The use of a discrete flavor symmetry to understand the mass hierarchies and mixing angles for all Standard Model fermions was pursued in Ref. [5,6]. Other authors have employed non-Abelian discrete flavor symmetries [7–9] that have both one and two dimensional representations. This approach is particularly well suited for addressing the supersymmetric flavor problem where the first and second generation of superpartners should have very similar flavor properties and thus belong in the same representation of the flavor group. In this paper we restrict our attention to Abelian discrete symmetries. (B) The phenomenological predictions for quark mass ratios and Cabibbo-Kobayashi-Maskawa (CKM) matrix elements can be retained, but the contributions arise from a smaller number of parameters. (C) The process of creating the baryon asymmetry of the Universe by having CP-violating asymmetric decays of heavy neutrinos can be greatly enhanced as a natural consequence of solving issue (A) above [10]. (D) One can potentially solve the supersymmetric flavor problem by suppressing certain entries in the Yukawa matrices. The mechanism works by aligning the quark mass matrices with

the squark mass-squared matrices [11], and one does not need to require that the first and second generation squarks are degenerate. The mixing matrix for the squark-quark-gluino couplings can be made close to a unit matrix, and the undesirable flavor-changing neutral currents (FCNCs) are suppressed. The required suppressions are not possible with a $U(1)$ symmetry.

During the last few years, there has been great interest in using new continuous Abelian as well as discrete Abelian and non-Abelian symmetries in the minimal supersymmetric standard model (MSSM) to describe the experimental (phenomenological) data on the fermion masses and mixings [2], [12]- [23]. Superstring theories appear to have $U(1)$ symmetries and symmetries involving its discrete subgroups as a generic feature. If the $U(1)$ flavor symmetry is gauged then a general assignment of flavor charges to the fields will be anomalous. One can imagine the anomaly is canceled via the Green-Schwarz mechanism [24], and one must check whether the correct relations are satisfied. A convenient way to ensure that the flavor charges are amenable to cancellation is to have the flavor symmetry commute with the $SU(5)$ grand unified theory [20]. We present in this paper a model with a $U(1) \times Z_2$ flavor symmetry with at least four texture zeros (in the up and down quark Yukawa matrices) that commutes with the $SU(5)$ gauge group.

The paper is organized as follows. In Section II we briefly discuss flavor symmetries and how they can in principle account for the experimentally observed hierarchies in the quark masses and mixing angles. In Section III we discuss the possible role of a discrete component in the flavor symmetry. Section IV then lists the phenomenological requirements that must be met in the quark sector of the standard model. In Section V a particular model for which the flavor symmetry commutes with an $SU(5)$ grand unified symmetry is presented, and the phenomenology is extended to include the leptons. The consequences for neutrino oscillations and the charged lepton masses are discussed. Finally we present our conclusions in Section VI.

II. FLAVOR SYMMETRIES

The hierarchical structure of the fermion mass matrices hints that there may be a spontaneously broken family symmetry responsible for the suppression of Yukawa couplings. In this paper we employ supersymmetric Abelian horizontal symmetries. These flavor symmetries allow the fermion mass and mixing hierarchies to be naturally generated from nonrenormalizable terms in the effective low-energy theory.

The idea is quite simple and easily implemented [12]. There is some field S which is charged under a $U(1)$ family symmetry, and without loss of generality, we can assume that its charge is -1. There are terms contributing to effective Yukawa couplings for the quarks,

$$Q_i \bar{d}_j H_d \left(\frac{S}{\Lambda_L} \right)^{m_{ij}} + Q_i \bar{u}_j H_u \left(\frac{S}{\Lambda_L} \right)^{n_{ij}}, \quad (2)$$

and the integer exponents m_{ij} and n_{ij} are easily calculated in terms of the horizontal symmetry charges of the quark and Higgs fields. The scale, Λ_L , where massive states are integrated out of the fundamental theory to produce an effective theory, is assumed to be larger than the vev, $\langle S \rangle$ of the singlet scalar field so the parameter $\langle S \rangle / \Lambda_L$ is a small one. We

henceforth require the Higgs fields to be uncharged under the $U(1)$ family symmetry, then the exponent m_{ij} is just the sum of the horizontal charge of the fields Q_i and \bar{d}_j . The hierarchy is generated from terms in the superpotential that carry integer charges $m_{ij}, n_{ij} \geq 0$. If we call the small breaking parameter $\langle S \rangle / \Lambda_L \sim \lambda$, then the generated terms for say the down quark Yukawa matrix will be of order $\lambda^{m_{ij}}$. We will restrict our attention in this paper to flavor charges for the Standard Model fields that are non-negative. Here texture zeros refer to Yukawa matrix elements that can be replaced by an exact zero without affecting the leading order (in the small parameter λ) results for the mass eigenvalues and mixing angles. An analysis of the possible approaches to explaining the neutrino masses and mixings using $U(1)$ symmetries only is given in Ref. [25].

In models whose flavor symmetry contains two distinct components ($U(1) \times U(1)$, $U(1) \times Z_m$, etc.) one introduces [4] two singlet scalars, S_1 and S_2 , with horizontal charges

$$S_1(-1, 0), \quad S_2(0, -1), \quad (3)$$

which in general can have different vacuum expectation values $\langle S_1 \rangle$ and $\langle S_2 \rangle$. These can be related to a common expansion parameter λ by setting

$$\frac{\langle S_1 \rangle}{\Lambda_L} \sim \lambda^\beta, \quad \frac{\langle S_2 \rangle}{\Lambda_L} \sim \lambda^\alpha. \quad (4)$$

In the following we identify λ as the Cabibbo angle, and take $\beta = 1$. In general, one can take $\alpha \neq 1$, but for our explicit models we assume $\alpha = 1$. The contributions to the Yukawa matrices arise from flavor invariant terms in

$$Q_i \bar{d}_j H_d \left(\frac{S_1}{\Lambda_L} \right)^{m_{ij}} \left(\frac{S_2}{\Lambda_L} \right)^{p_{ij}} + Q_i \bar{u}_j H_u \left(\frac{S_1}{\Lambda_L} \right)^{n_{ij}} \left(\frac{S_2}{\Lambda_L} \right)^{q_{ij}}. \quad (5)$$

It should be understood that there are undetermined order one coefficients multiplying these terms, and we assume in this paper that these coefficients are sufficiently close to one so as not to influence the hierarchy, i.e. somewhat greater than λ and somewhat less than $1/\lambda$. Formulae for the Yukawa matrices for the quarks and charged leptons as well as mass matrices for the neutrinos that follow from the Froggatt-Nielsen mechanism are given in the Appendix.

If the flavor symmetry is $U(1)$ then there is a charge assignment in the quark sector that satisfies all the phenomenological requirements detailed in Section IV below. This solution was obtained by many authors [18,26,27]. The up and down quark Yukawa matrices are

$$\mathbf{U} \sim \begin{pmatrix} \lambda^8 & \lambda^5 & \lambda^3 \\ \lambda^7 & \lambda^4 & \lambda^2 \\ \lambda^5 & \lambda^2 & 1 \end{pmatrix}, \quad \mathbf{D} \sim \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda & 1 & 1 \end{pmatrix}. \quad (6)$$

The model we present in this paper will give the same phenomenological predictions as Eq. (6), but the discrete symmetry will suppress certain entries in comparison to the $U(1)$ flavor symmetry pattern shown in Eq. (6). After including a discrete component to the flavor symmetry, a different $SU(5)$ grand unified model can be constructed (see Section V).

III. DISCRETE ABELIAN GROUPS

In this section we discuss the two possible texture patterns for a 2×2 matrix, and then show how to put these 2×2 blocks together to form the realistic case of texture patterns for three generations.

A. Suppressing in 2×2 blocks

When the flavor symmetry is $U(1)$, there is a sum rule among the exponents in any 2×2 block. For example, the up quark Yukawa matrix necessarily has the relationship,

$$n_{ii} + n_{jj} - n_{ij} - n_{ji} = 0 \quad (7)$$

between the exponents, $n_{ij} \equiv q_i + u_j$. The Yukawa matrices in Eq. (6) obey this rule, for example. However, these relationships between elements of the Yukawa matrices can be avoided if the flavor symmetry has a Z_m component. We can illustrate this with a simple example with a Z_2 symmetry: Consider two generations with Z_2 flavor charges as follows,

$$Q_L : \quad q_i^Z = (0, 1) , \quad \bar{u}_R : \quad u_i^Z = (0, 1) , \quad i = 1, 2 , \quad (8)$$

where the first number for each field gives the charge for the first generation and the second number gives the charge for the second generation. Then performing the Z_2 arithmetic in constructing the contribution to the Yukawa matrices yields, in general,

$$\begin{pmatrix} \lambda^{[q_1^Z + u_1^Z]} & \lambda^{[q_1^Z + u_2^Z]} \\ \lambda^{[q_2^Z + u_1^Z]} & \lambda^{[q_2^Z + u_2^Z]} \end{pmatrix} . \quad (9)$$

We use brackets around the exponents, $[\]$, to denote that we are modding out by two according to the Z_2 addition rules. In the case of the particular choice of charges in Eq. (8) and taking $\langle S_2 \rangle / \Lambda_L \sim \lambda$

$$\begin{pmatrix} \lambda^0 & \lambda^1 \\ \lambda^1 & \lambda^0 \end{pmatrix} , \quad (10)$$

So this set of charges yields a Yukawa matrix that does not satisfy the rule in Eq. (7). If one adds in nontrivial contributions from the $U(1)$ part of the flavor symmetry, one sees that the off-diagonal entries in Eq. (10) are suppressed relative to the expectation from Eq. (7). For example, assume that the fields have (in addition to the Z_2 assignments in Eq. (10)) the $U(1)$ charge assignments

$$Q_L : \quad q_i = (3, 0) , \quad \bar{u}_R : \quad u_i = (1, 0) , \quad i = 1, 2 , \quad (11)$$

which, in general, give a contribution to the Yukawa matrices

$$\begin{pmatrix} \lambda^{q_1 + u_1} & \lambda^{q_1 + u_2} \\ \lambda^{q_2 + u_1} & \lambda^{q_2 + u_2} \end{pmatrix} . \quad (12)$$

The particular choice of charges in Eq. (11) together with taking $\langle S_1 \rangle / \Lambda_L \sim \lambda$ yields the contribution to the Yukawa matrices from the $U(1)$ charges of the form

$$\begin{pmatrix} \lambda^4 & \lambda^3 \\ \lambda^1 & \lambda^0 \end{pmatrix}. \quad (13)$$

It should be clear from Eq. (5) that the overall contribution to the Yukawa matrix is the product of the contribution from the Z_2 charges in Eq. (10) and the contribution from the $U(1)$ charges in Eq. (13) for each element of the Yukawa matrix. For the example here we get the following result for the $U(1) \times Z_2$ flavor symmetry,

$$\mathbf{U} \sim \begin{pmatrix} \lambda^4 & \lambda^4 \\ \lambda^2 & \lambda^0 \end{pmatrix}. \quad (14)$$

So one sees that the off-diagonal entries are suppressed relative to the expectation in Eq. (13), and it is not difficult to convince oneself that this suppression comes entirely from the Z_2 part of the flavor symmetry.

The relevance of the above example to the present paper is the following. We are interested in determining the phenomenological predictions of the Yukawa matrices and then comparing them to the experimental data. This requires that we diagonalize the Yukawa matrices to determine the eigenvalues (masses) and the mixing angles (CKM elements) as explained in the Appendix. In the example we arrived at the Yukawa matrix in Eq. (14), for which it is immediately clear that the eigenvalues are of order λ^0 and λ^4 , while the mixing angles in V_u^L and V_u^R (see Eq. (58)) are of order λ^4 and λ^2 , respectively. So one notes that the left-handed mixing angle is suppressed by λ^2 in comparison to the expectation from a $U(1)$ symmetry alone that gives the same mass eigenvalues, namely

$$\begin{pmatrix} \lambda^4 & \lambda^2 \\ \lambda^2 & \lambda^0 \end{pmatrix}. \quad (15)$$

One can compare this simple 2×2 example with the second and third generations of Eq. (6). We denote this suppression in the following way,

$$\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}. \quad (16)$$

We say that the Yukawa matrix has texture zeros in the off-diagonal positions. A texture zero defined in this way is not a true zero, but is negligible to the leading order in the small parameter λ as far as the mass eigenvalues and the left-handed mixing (diagonalization) angles are concerned. The right-handed mixing angle is *not* suppressed but this affects neither the CKM mixing angles nor the mass eigenvalues. The physical observables are the elements of the CKM matrix, Eq. (59), and involve contributions from the diagonalization of the down-Yukawa matrix as well. So if there is a contribution from the diagonalization of down-Yukawa matrix of order λ , the contribution from the up-Yukawa matrix will be negligible in comparison. In this case we promote the texture zero to a true phenomenological zero: the contribution from the off-diagonal elements of the up-quark Yukawa matrix does not contribute to the determination of any physical observable (quark mass or CKM element) to leading order in the expansion in the small parameter λ .

We can also engineer a suppression along the diagonal elements of a Yukawa matrix. This case is somewhat trickier than the previous case, so we proceed now to present another example in the case of just two generations: For the Z_2 charges, consider the assignment

$$Q_L : \quad q_i^Z = (1, 0) , \quad \bar{d}_R : \quad d_i^Z = (0, 1) , \quad i = 1, 2 . \quad (17)$$

and, for the $U(1)$ charges, make the assignment

$$Q_L : \quad q_i = (3, 0) , \quad \bar{d}_R : \quad d_i = (1, 0) , \quad i = 1, 2 . \quad (18)$$

Then one obtains

$$\begin{pmatrix} \lambda^1 & \lambda^0 \\ \lambda^0 & \lambda^1 \end{pmatrix} . \quad (19)$$

for the contribution from the Z_2 charges and

$$\begin{pmatrix} \lambda^4 & \lambda^3 \\ \lambda^1 & \lambda^0 \end{pmatrix} , \quad (20)$$

from the $U(1)$ sector. The contributions from the full $U(1) \times Z_2$ symmetry give the Yukawa matrix,

$$\mathbf{D} \sim \begin{pmatrix} \lambda^5 & \lambda^3 \\ \lambda^1 & \lambda^1 \end{pmatrix} . \quad (21)$$

The eigenvalues of this matrix are λ^0 and λ^2 and the mixing angle for the left-handed diagonalization matrix is of order λ^2 . So one can interpret the [1,1] entry of order λ^5 as being phenomenologically irrelevant to leading order in powers of λ and the texture zero pattern is

$$\begin{pmatrix} 0 & X \\ X & X \end{pmatrix} . \quad (22)$$

Note that the [2,2] entry is not phenomenologically irrelevant, and is still denoted by X .

Having generated the Yukawa matrices \mathbf{U} and \mathbf{D} in Eqs. (14) and (21), we can account for phenomenological requirements (the full set of experimental data for fermion masses and mixings is given in the next section),

$$\frac{m_c}{m_t} \sim \lambda^4 , \quad \frac{m_s}{m_b} \sim \lambda^2 , \quad |V_{cb}| \sim \lambda^2 . \quad (23)$$

The mixing angles for \mathbf{U} are $\sin \theta_L^u \sim \lambda^4$ and $\sin \theta_R^u \sim \lambda^2$, while for \mathbf{D} they are $\sin \theta_L^d \sim \lambda^2$ and $\sin \theta_R^d \sim \lambda^0$. The leading order contribution to $|V_{cb}|$ according to Eq. (59) is then given entirely by $\sin \theta_L^d \sim \lambda^2$ since $\sin \theta_L^u$ is suppressed by a relative factor of λ^2 . The mass eigenvalue ratios in Eq. (23) are properly accounted for. So phenomenologically viable Yukawa matrices can be found with texture zeros, and these zeros reduce the number of unknown order one coefficients that contribute to masses and mixing angles at the leading order in λ .

An important feature of the $U(1) \times Z_2$ flavor symmetry is that one can achieve different leading order contributions to the left-handed mixing angles in V_u^L and V_d^L , as shown in the above example. In a model with a $U(1)$ symmetry, these mixing angles are determined entirely by the charges Q_L (and not \bar{u}_R and \bar{d}_R). So the presence of the Z_2 symmetry allows one to suppress the contribution to the CKM mixings from either the \mathbf{U} or the \mathbf{D} matrix.

It is not difficult to generalize the discussion to an arbitrary Z_m . The exponents of λ are given by Eq. (9), and the conditions satisfied by the Z_m charges that lead to texture suppressions are

$$[q_i^Z + u_i^Z] + [q_j^Z + u_j^Z] - [q_i^Z + u_j^Z] - [q_j^Z + u_i^Z] = \pm m, \quad m \geq 2, \quad (24)$$

where the case $+m$ results in a suppression of the diagonal entries of the 2×2 matrix, and the case $-m$ results in a suppression of the off-diagonal entries. We remind the reader that the square brackets in Eq. (24) indicate a modding by the integer m .

B. Extending Z_m -induced suppressions to 3×3 matrices

In the last subsection examples of how to suppress entries in a 2×2 Yukawa matrix were presented. One can extend this result to the three generation case by considering 2×2 blocks. One has three such blocks in the case of three generations: namely the [2-3], [1-3] and the [1-2] blocks. One can build up a texture pattern for a 3×3 matrix by placing zeros in the desired positions of these 2×2 blocks. As demonstrated in the last subsection, in each 2×2 block, one can have either a texture zero in the off-diagonal or in a diagonal position, but not both at the same time. As an example consider the matrix:

$$\begin{pmatrix} 0 & 0 & X \\ 0 & X & X \\ X & X & X \end{pmatrix} \quad (25)$$

All the zeros cannot be obtained by assigning charges in the [1-2] block alone, since this would require the zeros to be in both the diagonal and off-diagonal positions. However this texture pattern can be obtained by assigning the zero on the diagonal to the [1-3] block, while off-diagonal zeros can be assigned to the [1-2] block. As in the case of only two generations, one can obtain the texture pattern by considering only the Z_m component of the flavor symmetry. One can obtain the required texture in Eq. (25) when $m = 3$ by the following assignment of Z_3 charges

$$Q_L : \quad q_i^Z = (2, 0, 1), \quad \bar{u}_R : \quad u_i^Z = (2, 0, 1). \quad i = 1, 2, 3, \quad (26)$$

The contribution to the 3×3 matrix Yukawa matrix from this Z_3 charge assignment is

$$\begin{pmatrix} \lambda^1 & \lambda^2 & \lambda^0 \\ \lambda^2 & \lambda^0 & \lambda^1 \\ \lambda^0 & \lambda^1 & \lambda^2 \end{pmatrix}, \quad (27)$$

The $U(1)$ contributions have not been included yet in Eq. (27). While the [2-3] block does not have a suppressing pattern, the suppression in the [1-3] block suppresses the diagonal element [1,1]. Finally, the [1-2] block suppresses the off-diagonal [1,2] and [2,1] elements.

Continuing with our example, if we assign the $U(1)$ flavor charges

$$Q_L : \quad q_i = (6, 3, 0), \quad \bar{u}_R : \quad u_i = (6, 3, 0). \quad i = 1, 2, 3, \quad (28)$$

to the quark fields, we obtain the following contribution to the up-type Yukawa matrix,

$$\begin{pmatrix} \lambda^{12} & \lambda^9 & \lambda^6 \\ \lambda^9 & \lambda^6 & \lambda^3 \\ \lambda^6 & \lambda^3 & 1 \end{pmatrix}. \quad (29)$$

Putting the contributions from both components of the $U(1) \times Z_3$ flavor symmetry together gives the following up-type Yukawa matrix (after dropping an overall factor of λ^2 which is irrelevant as far as the hierarchy is concerned),

$$\mathbf{U} \sim \begin{pmatrix} \lambda^{11} & \lambda^9 & \lambda^4 \\ \lambda^9 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix}. \quad (30)$$

One can always diagonalize matrices arises from Abelian flavor symmetries of the type described here in stages [28], by diagonalizing the [2-3] block, followed by the [1-3] block, and finally diagonalizing the [1-2] block. The diagonalization in the [2-3] block does not produce any texture zero because $(\lambda^2)(\lambda^2)/(1)(\lambda^4)$ as in Eq. (7). Each order one coefficient in the [2-3] block plays a role in determining the leading order diagonalization of that block. However in the diagonalization of the [1-3] block, one notices that [1,1] element (λ^{11}) is suppressed by a factor of order λ^3 compared to the product of the [1,3] and [3,1] elements. So to leading order in an expansion in λ , the diagonalization of the matrix in Eq. (30) is the same as a matrix where the λ^{11} element is replaced with zero (and we call such an entry a texture zero). So we have the following matrix whose diagonalization is equivalent to leading order to the original matrix

$$\mathbf{U} \sim \begin{pmatrix} 0 & \lambda^9 & \lambda^4 \\ \lambda^9 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix}. \quad (31)$$

Finally we must determine if any of the elements in the [1-2] block are suppressed. Suppose the diagonalization has been performed in the [2-3] and [1-3] blocks. Then the matrix has the form

$$\begin{pmatrix} \lambda^8 & \lambda^9 & 0 \\ \lambda^9 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (32)$$

The λ^8 entry would be generated in the [1,1] element (and the λ^{11} element can be neglected in comparison, as described above). So a subsequent diagonalization of the [1-2] block indicates that texture zeros occur in the off-diagonal elements as

$$\mathbf{U} \sim \begin{pmatrix} 0 & 0 & \lambda^4 \\ 0 & \lambda^4 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{pmatrix}. \quad (33)$$

In other words, to leading order in λ the diagonalization of the first matrix in Eq. (30) is the same as the diagonalization of the matrix in Eq. (33).

By proceeding in this way, one can systematically construct all possible matrices with texture zeros in the desired positions. The task then is to combine a texture pattern for the up-type Yukawa matrix with another texture pattern for the down-type Yukawa matrix, and check whether all the phenomenological requirements can be satisfied. We now turn to the experimental data for the quark and lepton masses and mixing angles.

IV. PHENOMENOLOGICAL REQUIREMENTS IN THE QUARK SECTOR

If one must satisfy the phenomenological constraints with positive flavor charges, then the Eq. (6) is the solution that results from a $U(1)$ flavor symmetry. Using a $U(1) \times Z_m$ flavor symmetry instead will change the exponents by adding m in certain elements. The relevant equations for the CKM matrix elements that are valid for this category of matrices are [6],

$$|V_{us}| = \left(\frac{d_{12}}{\tilde{d}_{22}} - \frac{d_{13}d_{32}}{\tilde{d}_{22}} \right) - \left(\frac{u_{12}}{\tilde{u}_{22}} - \frac{u_{13}u_{32}}{\tilde{u}_{22}} \right), \quad (34)$$

$$|V_{cb}| = d_{23} + d_{22}d_{32}^* - u_{23}, \quad (35)$$

$$|V_{ub}| = (d_{13} + d_{12}d_{32}^* - u_{13}) - \left(\frac{u_{12}}{\tilde{u}_{22}} - \frac{u_{13}u_{32}}{\tilde{u}_{22}} \right) (d_{23} + d_{22}d_{32}^* - u_{23}), \quad (36)$$

$$|V_{td}| = -(d_{13} + d_{12}d_{32}^* - u_{13}) + \left(\frac{d_{12}}{\tilde{d}_{22}} - \frac{d_{13}d_{32}}{\tilde{d}_{22}} \right) (d_{23} + d_{22}d_{32}^* - u_{23}), \quad (37)$$

where $d_{ij} = \mathbf{D}_{ij}/\mathbf{D}_{33}$ and $\tilde{d}_{22} = d_{22} - d_{23}d_{32}$ and $\tilde{u}_{22} = u_{22} - u_{23}u_{32}$. It is understood that there will in general be relative phases between the terms on the right hand sides of Eqs. (34)-(37), which are the correct forms to evaluate the leading orders for Yukawa matrices of the form considered in this paper.

Taking the expansion parameter to be the Cabibbo angle, $\lambda = |V_{us}|$, then the experimental constraints [29]

$$|V_{us}| = 0.2196 \pm 0.0023, \quad |V_{cb}| = 0.0395 \pm 0.0017, \quad \left| \frac{V_{ub}}{V_{cb}} \right| = 0.08 \pm 0.02, \quad (38)$$

on the CKM matrix can be identified in terms of powers of λ by the following¹,

$$|V_{us}| \sim \lambda, \quad |V_{cb}| \sim \lambda^2, \quad |V_{ub}| \sim \lambda^3 - \lambda^4, \quad \left| \frac{V_{ub}}{V_{cb}} \right| \sim \lambda - \lambda^2. \quad (39)$$

We consider a model of Yukawa matrices to describe the experimental data satisfactorily if the leading order contribution to the CKM elements agrees with Eq. (39). For $|V_{ub}|$ and $|V_{ub}/V_{cb}|$ we accept two values for the exponent of the leading contribution. The constraint

¹There are renormalization scaling factors that relate the experimental data at the electroweak scale, Eq. (38), to the relationships at the high scale [30].

on $|V_{ub}/V_{cb}|$ can be expressed in a stronger way at 90% confidence level as $0.25\lambda - 0.5\lambda$. One also has a constraint on the CKM elements from $B_d^0 - \bar{B}_d^0$ mixing [29],

$$|V_{tb}^* V_{td}| = 0.0084 \pm 0.0018, \quad (40)$$

which implies that

$$|V_{td}| \sim \lambda^3. \quad (41)$$

The eigenvalues of the Yukawa matrices are constrained by the following requirements from experimental observations

$$\frac{m_c}{m_t} \sim \lambda^4, \quad \frac{m_u}{m_c} \sim \lambda^4, \quad \frac{m_s}{m_b} \sim \lambda^2, \quad \frac{m_d}{m_s} \sim \lambda^2. \quad (42)$$

These phenomenological requirements will be used in the next section to constrain the Yukawa matrix patterns that can successfully reproduce the experimental data.

V. GRAND UNIFIED MODEL

In this section we derive an assignment of charges in $U(1) \times Z_2$ that has the maximum number of texture suppressions (four) that is consistent with a $SU(5)$ grand unified symmetry. Since the flavor symmetry is required to commute with $SU(5)$, this means that there must be a common flavor charge assignment for all particles in each multiplet of $SU(5)$. We restrict our attention to the case of a Z_2 symmetry, since (as described earlier) it is the only possible Z_m symmetry that can reproduce a hierarchy in neutrino masses of order λ^2 [2].

Firstly, we have found that all the solutions from the $U(1) \times Z_2$ flavor symmetry that satisfy the quark sector phenomenology have the following property: the $[3,2]$ entry and the $[3-3]$ entry of the down quark Yukawa matrix, \mathbf{D} , are the same order of magnitude. If the flavor symmetry is embedded in a grand unified model, the charged lepton Yukawa matrix will be given by the transpose of \mathbf{D} . Then the feature of Eq. (6), that the right-handed mixing matrix that diagonalizes the down-quark Yukawa matrix, \mathbf{D} , is of order one in the $[2-3]$ block, is retained. This has the important consequence that, if the lepton charges are related to the down quark charges by a grand unified theory, then the charged lepton matrix will require a large mixing between in the $[2-3]$ block to diagonalize it. This results in a large mixing between the second and third generation of neutrinos, and can naturally explain how the atmospheric neutrino mixing can be large (order one) while the quark mixing between the second and third generations, $|V_{cb}|$, can be small (order λ^2). This has been called the “lopsided” solution to the producing the required atmospheric neutrino mixing in grand unified models [31]. This occurs in all the models necessarily after applying the phenomenological requirements $|V_{cb}| \sim \lambda^2$ and $m_s/m_b \sim \lambda^2$.

In models in which the $U(1)$ flavor symmetry is gauged and anomalous, one can imagine the anomaly is canceled via the Green-Schwarz mechanism [24]. A convenient way to ensure that the flavor charges are amenable to cancellation is to have the flavor symmetry commute

with the $SU(5)$ grand unified theory². In the traditional $SU(5)$ grand unified theory, the fields Q_L and \bar{u}_R are assigned to the **10** representation, and the \bar{d}_R is assigned to the **5**^{*} representation. We have found a texture pattern for the up and down quark Yukawa matrices with four texture zeros for which the flavor symmetry quantum number assignment commutes with an $SU(5)$ grand unified gauge group. This texture pattern yields

$$\mathbf{U} \sim \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix} \quad \mathbf{D} \sim \begin{pmatrix} 0 & 0 & \lambda^3 \\ 0 & \lambda^2 & \lambda^2 \\ \lambda & 1 & 1 \end{pmatrix} \quad \begin{array}{l} Q_L : (4, 1) \quad (2, 0) \quad (0, 0) \\ \bar{u}_R : (4, 1) \quad (2, 0) \quad (0, 0) \\ \bar{d}_R : (2, 0) \quad (1, 0) \quad (0, 1) \end{array} . \quad (43)$$

This assignment has common $U(1) \times Z_2$ flavor symmetry quantum numbers for the Q_L and \bar{u}_R fields in the **10**, and a systematic search reveals that no other texture pattern with four or more texture zeros satisfies this property. Finding an assignment for which the flavor symmetry commutes with $SU(5)$ allows us to assign flavor charges to the rest of the $SU(5)$ multiplets, namely the charged leptons and neutrinos.

The texture pattern given by Eq. (43) has the following feature: The CKM mixing $|V_{cb}|$ arises from contributions of order λ^2 from the diagonalizations of both the \mathbf{U} and \mathbf{D} Yukawa matrices. All other CKM mixing angles ($|V_{us}|$, $|V_{ub}|$ and $|V_{td}|$) arise solely from the \mathbf{D} Yukawa matrix.

Given the quantum number assignment in Eq. (43), we can extend the model to encompass the leptons. The field \bar{e}_R fills out the **10** representation, and the left-handed lepton doublet, L_L , fills out the **5**^{*} representation, so they should have the quantum number assignments

$$\begin{array}{lll} i = & 1 & 2 & 3 \\ \bar{e}_R : & (4, 1) & (2, 0) & (0, 0) , \\ L_L : & (2, 0) & (1, 0) & (0, 1) \end{array} \quad (44)$$

These assignments dictated by the Eq. (43) can be compared against constraints obtained from experiment for masses and mixings in the lepton sector. The first phenomenological constraints we consider involve the charged leptons. Using the $U(1) \times Z_2$ quantum numbers in Eq. (44), one immediately obtains the charged lepton Yukawa matrix (see the Appendix for formulae),

$$m_{\ell\pm} \sim \begin{pmatrix} \lambda^7 & \lambda^4 & \lambda^2 \\ \lambda^6 & \lambda^3 & \lambda \\ \lambda^4 & \lambda^3 & \lambda \end{pmatrix} v_1 . \quad (45)$$

As desired the [2,3] and [3,3] elements are the same order of magnitude. This yields the mass ratios

²The mixed Standard Model- $U(1)$ anomalies can be canceled entirely by the Green-Schwarz mechanism if the $U(1)$ charges X satisfy the relations $tr(XT_aT_b) \propto tr(T_aT_b)$ and $tr(X^2Y) = 0$ where T_a are the Standard Model generators. These relations are satisfied automatically if the $U(1)$ charges respect the $SU(5)$ symmetry.

$$\frac{m_\mu}{m_\tau} \sim \lambda^2, \quad \frac{m_e}{m_\tau} \sim \lambda^4, \quad (46)$$

which are consistent with the experimental constraints after including renormalization group scaling [30]³.

Next consider the light neutrino mass matrix. There are two possibilities that were discussed previously in Ref. [6]. First the light neutrino mass matrix might not have suppressed entries arising from the Z_2 component of the flavor symmetry, in which case the light neutrino mass matrix is simply given by Eq. (68), where L_i in this case is simply the sum of the $U(1)$ and Z_2 quantum numbers of the relevant lepton doublet field, L_L . For the charge assignments in Eq. (44), this gives

$$m_\nu \sim \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda^2 \end{pmatrix} \frac{v_2^2}{\Lambda_L}. \quad (47)$$

The remaining constraints on leptons involve the neutrino masses and mixings. The most interesting aspect of the neutrino data is that the atmospheric neutrino mixing appears to be large, perhaps even maximal. As mentioned earlier, it is difficult to understand a hierarchical pattern for the neutrino masses, since large mixing should result when the neutrino masses are of roughly the same order of magnitude. The Super-Kamiokande data [1] suggest that

$$\Delta m_{23}^2 \sim 2.2 \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{23}^\nu \sim 1, \quad (48)$$

where the subscripts indicate the generations of neutrinos involved in the mixing (we assume the mixing is between ν_μ and ν_τ , and not some sterile neutrino).

The solar neutrino flux can be explained by one of three distinct solutions. Two of these involve matter-enhanced oscillation (MSW), while the third involves vacuum oscillations (VO). The two MSW solutions are differentiated by the size of the mixing angle, so one is usually called the small mixing angle (SMA) solution, and the other is called the large mixing angle (LMA) solution. The values required for the mixing parameters in each of these three cases are shown in the table below.

	$\Delta m_{1x}^2 [eV^2]$	$\sin^2 2\theta_{1x}$
MSW(SMA)	5×10^{-6}	6×10^{-3}
MSW(LMA)	2×10^{-5}	0.8
VO	8×10^{-11}	0.8

The MSW solutions can be obtained with a Z_2 horizontal symmetry [2,6,10]. If the neutrino masses are arranged in a hierarchy, then the best fit to the data is

$$\frac{\Delta m_{12}^2}{\Delta m_{23}^2} \sim \lambda^4, \quad \sin \theta_{23}^\nu \sim \lambda^0, \quad (49)$$

³The largest scaling effect results from the additional running necessary to reach the muon and the electron mass scales so that one can relate the Yukawa couplings to the physical masses of the charged leptons. The scaling of the Yukawa coupling ratios themselves is negligibly small.

and either

$$\sin \theta_{12}^\nu \sim \lambda^2, \quad (50)$$

for the SMA solution, or

$$\sin \theta_{12}^\nu \sim \lambda^0, \quad (51)$$

for the LMA solution. If no Z_2 symmetry is operative, one gets a light neutrino mass as in Eq. (68), and if $L_2 = L_3$, there is no natural explanation for the hierarchy in the masses of m_{ν_μ} and m_{ν_τ} . As explained in Refs. [6,10], this can be remedied by assigning the right-handed neutrino fields $\bar{\nu}_{Ri}$ (singlets of $SU(5)$) the following Z_2 charges: (0,0,1). The particular $U(1)$ assignment for the fields $\bar{\nu}_{Ri}$ does not affect the light neutrino mass matrix. In this case, the [3,3] element of the m_ν matrix is enhanced by a factor λ^{-2} , giving

$$m_\nu \sim \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \frac{v_2^2}{\Lambda_L}, \quad (52)$$

for the charge assignment in Eq. (44).

The neutrino mixing matrix is

$$\begin{pmatrix} 1 & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix}. \quad (53)$$

The solar mixing angle is predicted to be of order λ , falling in between the optimal value for the LMA solution (λ^0) and the SMA solution (λ^2). Equation (6) yields a solar mixing angle of order λ^3 , so the presence of the Z_2 symmetry has the effect in the neutrino sector of enhance the mixing of the first generation to the second and third generations by a factor λ^{-2} . Several unknown order-one coefficients combine to produce the matrix in Eq. (52), so it is not necessarily inconsistent with the MSW solutions.

In the models described here, one can achieve alignment of the quark mass matrices and the squark mass-squared matrices by certain positioning of texture zeros in the quark Yukawa matrices. This alignment can solve the SUSY flavor problem by making it possible to simultaneously diagonalize the quark mass matrices and the quark-squark-gluino coupling, thereby avoiding the dangerous flavor-changing couplings. In particular, in the models we are discussing here, one can achieve this alignment if there are texture zeros in the down quark Yukawa matrix, \mathbf{D} , in the [1,2] and [2,1] elements, and in either the [1,3] or [3,2] elements, and in either the [1,2] or [3,1] elements. This is easily seen to be the case after a quick inspection of Eqs. (34)-(37): in this case the Cabibbo angle, $|V_{us}|$, arises to leading order solely in the up quark Yukawa matrix, \mathbf{U} . The texture patterns that achieve this alignment occur when the down-quark Yukawa matrix has texture zeros in the positions given by the patterns

$$\begin{pmatrix} X & 0 & 0 \\ 0 & 0 & X \\ 0 & X & X \end{pmatrix} \quad (54)$$

$$\begin{pmatrix} 0 & 0 & X \\ 0 & X & 0 \\ X & 0 & X \end{pmatrix} \quad (55)$$

which have the off-diagonal elements in the [1-2] block doubly suppressed. The off-diagonal suppression in the [1-3] block in the case of pattern Eq. (54) or the [2-3] block in the case of the pattern Eq. (55) need to be doubly suppressed, which is impossible. So one cannot achieve the quark-squark alignment in the context of a $U(1) \times Z_2$ flavor symmetry. On the other hand, one can employ the idea of supersymmetry breaking through an anomalous flavor symmetry [32,33] to the grand unified model presented in this section. One can obtain reasonable suppression of the flavor-changing effects provided the first and second generation sparticles are in the multi-TeV range [19–21].

VI. CONCLUSION

We have shown that if the fermion mass matrices are dictated by an Abelian family symmetry, one can obtain a phenomenologically successful texture pattern by employing additional Z_m horizontal symmetries. This four-texture zero model has a flavor symmetry that commutes with an $SU(5)$ grand unified theory with the usual assignment of particles to the $\mathbf{5}^*$ and $\mathbf{10}$ representations. When the quantum numbers are extended to the lepton sector, the charged lepton mass ratios were correctly predicted and a large mixing angle naturally arises to explain the atmospheric neutrino data. A mixing angle of order λ arises to explain the solar neutrino oscillation data.

Discrete flavor symmetries can suppress entries in the Yukawa matrices and offer the potential of a solution to the supersymmetric flavor problem. A judicious choice of texture zeros can render the quark mass matrices and the squark mass-squared matrices simultaneously diagonalizable, thereby eliminating some strongly constrained flavor-changing couplings. However, we find that this solution cannot be obtained in a model with a single Z_2 symmetry and satisfy all the other (masses and mixings) phenomenological requirements. However the quantum number assignments can be compatible through suppression of flavor-changing effects when supersymmetry breaking is mediated by the anomalous flavor symmetry.

APPENDIX

In this appendix we review the formulae for Yukawa and mass matrices that result from Abelian horizontal symmetries with and without a discrete component. Let the $U(1)$ quark charges be given by

$$\begin{array}{ccccccccc} Q_{L1} & Q_{L2} & Q_{L3} & \bar{u}_{R1} & \bar{u}_{R2} & \bar{u}_{R3} & \bar{d}_{R1} & \bar{d}_{R2} & \bar{d}_{R3} \\ q_1 & q_2 & q_3 & u_1 & u_2 & u_3 & d_1 & d_2 & d_3 \end{array}.$$

Then the up and down quark Yukawa matrices, \mathbf{U} and \mathbf{D} are given by⁴

$$\mathbf{U} \sim \begin{pmatrix} \lambda^{q_1+u_1} & \lambda^{q_1+u_2} & \lambda^{q_1+u_3} \\ \lambda^{q_2+u_1} & \lambda^{q_2+u_2} & \lambda^{q_2+u_3} \\ \lambda^{q_3+u_1} & \lambda^{q_3+u_2} & \lambda^{q_3+u_3} \end{pmatrix}, \quad \mathbf{D} \sim \begin{pmatrix} \lambda^{q_1+d_1} & \lambda^{q_1+d_2} & \lambda^{q_1+d_3} \\ \lambda^{q_2+d_1} & \lambda^{q_2+d_2} & \lambda^{q_2+d_3} \\ \lambda^{q_3+d_1} & \lambda^{q_3+d_2} & \lambda^{q_3+d_3} \end{pmatrix}. \quad (56)$$

⁴We use the notation $a \sim b$ to indicate that a and b are the same order in λ .

It is understood that these matrices have unknown coefficients multiplying each element. The contributions to each element arise from a different operator in Eq. (5), so they are in general independent of each other. Since these coefficients are not correlated there is no reason to expect the Yukawa matrices to have a zero eigenvalue.

To compare the predictions of flavor symmetries to these phenomenological constraints, one has to relate the CKM elements to the entries in the Yukawa matrices. The Yukawa matrices \mathbf{U} and \mathbf{D} can be diagonalized by biunitary transformations

$$\mathbf{U}^{\text{diag}} = V_u^L \mathbf{U} V_u^{R\dagger}, \quad (57)$$

$$\mathbf{D}^{\text{diag}} = V_d^L \mathbf{D} V_d^{R\dagger}. \quad (58)$$

The CKM matrix is then given by

$$V \equiv V_u^L V_d^{L\dagger}. \quad (59)$$

The left-handed transformation matrices V_u^L and V_d^L can be defined in terms of three successive rotations in the (2,3), (1,3) and (1,2) sectors. These rotation angles of the transformation matrices can be expressed in terms of the elements of the Yukawa matrices as follows [4,28]

$$s_{12}^u = \frac{u_{12}}{\tilde{u}_{22}} + \frac{u_{11}u_{21}^*}{|\tilde{u}_{22}^*|^2} - \frac{u_{13}(u_{32} + u_{23}^*u_{22})}{\tilde{u}_{22}} - \frac{u_{11}u_{31}^*(u_{23}^* + u_{32}u_{22}^*)}{|\tilde{u}_{22}^*|^2}, \quad (60)$$

$$s_{13}^u = u_{13} + u_{11}u_{31}^* + u_{12}(u_{32}^* + u_{22}^*u_{23}) + u_{11}u_{21}^*(u_{23} + u_{22}u_{32}^*), \quad (61)$$

$$s_{23}^u = u_{23} + u_{22}u_{32}^*, \quad (62)$$

where u_{ij} is the i, j th component of the up quark Yukawa matrix, $\mathbf{U}/(\mathbf{U})_{33}$, and $\tilde{u}_{22} = u_{22}u_{33} - u_{23}u_{32}$. There are corresponding expressions for the s_{ij}^d in terms of the components of the down quark Yukawa matrix, \mathbf{D} (which are slightly more complicated due to the fact that the (2,3) sector mixing in V_d^R might be of order one). Clearly contributions to the CKM matrix elements can come from a number of terms. In this paper we are interested in determining only the leading order contribution(s) to the CKM angles and the fermion masses.

Assume now that the lepton fields have charges under a $U(1)$ family symmetry

$$\begin{array}{ccccccccc} \bar{e}_{R1} & \bar{e}_{R2} & \bar{e}_{R3} & \ell_{L1} & \ell_{L2} & \ell_{L3} & \bar{\nu}_{R1} & \bar{\nu}_{R2} & \bar{\nu}_{R3} \\ E_1 & E_2 & E_3 & L_1 & L_2 & L_3 & \mathcal{N}_1 & \mathcal{N}_2 & \mathcal{N}_3 \end{array}. \quad (63)$$

All the flavor charges are non-negative so holomorphic zeros do not play a role. The only suppressed entries will arise because of a discrete component in the flavor symmetry via a mechanism described below.

Given lepton doublet charges L_i and right-handed neutrino charges \mathcal{N}_i one has the following pattern for the charged lepton matrix

$$m_{\ell^\pm} \sim \begin{pmatrix} \lambda^{L_1+E_1} & \lambda^{L_1+E_2} & \lambda^{L_1+E_3} \\ \lambda^{L_2+E_1} & \lambda^{L_2+E_2} & \lambda^{L_2+E_3} \\ \lambda^{L_3+E_1} & \lambda^{L_3+E_2} & \lambda^{L_3+E_3} \end{pmatrix} v_1, \quad (64)$$

and for the neutrino Dirac mass matrix

$$m_D \sim \begin{pmatrix} \lambda^{L_1+\mathcal{N}_1} & \lambda^{L_1+\mathcal{N}_2} & \lambda^{L_1+\mathcal{N}_3} \\ \lambda^{L_2+\mathcal{N}_1} & \lambda^{L_2+\mathcal{N}_2} & \lambda^{L_2+\mathcal{N}_3} \\ \lambda^{L_3+\mathcal{N}_1} & \lambda^{L_3+\mathcal{N}_2} & \lambda^{L_3+\mathcal{N}_3} \end{pmatrix} v_2 , \quad (65)$$

We have defined there the VEVs of the Higgs coupling to the down- and up-type quarks to be v_1 and v_2 , and one usually defines $\tan \beta = v_2/v_1$. To determine the neutrino mixing angles one rotates to a basis where the charged lepton matrix is diagonal. This will give a contribution to the mixing in the light neutrino species. The relevant mixing contributing to atmospheric neutrino oscillations comes from the right hand side of the charge lepton matrix, $\lambda^{L_2+E_3}/\lambda^{L_3+E_3}$.

The Majorana mass matrix is obtained from the charges of the right-handed neutrino flavor charges \mathcal{N}_i and a heavy scale we label as Λ_L ,

$$M_N \sim \begin{pmatrix} \lambda^{2\mathcal{N}_1} & \lambda^{\mathcal{N}_1+\mathcal{N}_2} & \lambda^{\mathcal{N}_1+\mathcal{N}_3} \\ \lambda^{\mathcal{N}_1+\mathcal{N}_2} & \lambda^{2\mathcal{N}_2} & \lambda^{\mathcal{N}_2+\mathcal{N}_3} \\ \lambda^{\mathcal{N}_1+\mathcal{N}_3} & \lambda^{\mathcal{N}_2+\mathcal{N}_3} & \lambda^{2\mathcal{N}_3} \end{pmatrix} \Lambda_L . \quad (66)$$

Then one obtains the following form for the light neutrino mass matrix via the see-saw formula

$$m_\nu = m_D \frac{1}{M_N} m_D^T , \quad (67)$$

where m_D is the neutrino Dirac mass matrix. Then [2,25],

$$m_\nu \sim \begin{pmatrix} \lambda^{2L_1} & \lambda^{L_1+L_2} & \lambda^{L_1+L_3} \\ \lambda^{L_1+L_2} & \lambda^{2L_2} & \lambda^{L_2+L_3} \\ \lambda^{L_1+L_3} & \lambda^{L_2+L_3} & \lambda^{2L_3} \end{pmatrix} \frac{v_2^2}{\Lambda_L} . \quad (68)$$

If $L_2 = L_3$ one can obtain $\mathcal{O}(1)$ mixing in the 2-3 sector [3]. On the other hand, one fails to get a mass hierarchy between the second and third generation, since the two mass eigenvalues for the second and third generations are both of order λ^{2L_3} .

A discrete Abelian family symmetry can be employed to enhance or suppress masses and mixing angle relative to the predictions obtained when the family symmetry is the continuous $U(1)$ symmetry, and this idea was pursued further in specific models [5,6]. The discrete Z_m symmetry can result in the enhancement of entries in the light neutrino mass matrix [2], and this enhancement is compatible with the neutrino seesaw mechanism [10,6]. For example, if the $U(1)$ quantum numbers in Eq. (63) are replaced by $U(1) \times Z_2$ quantum numbers, $L_3 \rightarrow (L_3 - 1, 1)$ and $\mathcal{N}_3 \rightarrow (\mathcal{N}_3 - 1, 1)$ so that the charges for the lepton fields are

$$\begin{matrix} \bar{e}_{R1} & \bar{e}_{R2} & \bar{e}_{R3} & \ell_{L1} & \ell_{L2} & \ell_{L3} & \bar{\nu}_{R1} & \bar{\nu}_{R2} & \bar{\nu}_{R3} \\ (E_1, 0) & (E_2, 0) & (E_3, 0) & (L_1, 0) & (L_2, 0) & (L_3 - 1, 1) & (\mathcal{N}_1, 0) & (\mathcal{N}_2, 0) & (\mathcal{N}_3 - 1, 1) \end{matrix} . \quad (69)$$

then one finds that

$$M_N \sim \begin{pmatrix} \lambda^{2\mathcal{N}_1} & \lambda^{\mathcal{N}_1+\mathcal{N}_2} & \lambda^{\mathcal{N}_1+\mathcal{N}_3} \\ \lambda^{\mathcal{N}_1+\mathcal{N}_2} & \lambda^{2\mathcal{N}_2} & \lambda^{\mathcal{N}_2+\mathcal{N}_3} \\ \lambda^{\mathcal{N}_1+\mathcal{N}_3} & \lambda^{\mathcal{N}_2+\mathcal{N}_3} & \lambda^{2\mathcal{N}_3-2} \end{pmatrix} \Lambda_L , \quad (70)$$

so that

$$(M_N)^{-1} \sim \begin{pmatrix} \lambda^{-2\mathcal{N}_1} & \lambda^{-\mathcal{N}_1-\mathcal{N}_2} & \lambda^{-\mathcal{N}_1-\mathcal{N}_3+2} \\ \lambda^{-\mathcal{N}_1-\mathcal{N}_2} & \lambda^{-2\mathcal{N}_2} & \lambda^{-\mathcal{N}_2-\mathcal{N}_3+2} \\ \lambda^{-\mathcal{N}_1-\mathcal{N}_3+2} & \lambda^{-\mathcal{N}_2-\mathcal{N}_3+2} & \lambda^{-2\mathcal{N}_3+2} \end{pmatrix} \Lambda_L^{-1} . \quad (71)$$

So the effect of the discrete symmetry in our case is to enhance the 3-3 entry of the M_N matrix, and thereby alter the results for the third row and the third column on the inverse matrix, $(M_N)^{-1}$. The 3-3 component of the neutrino Dirac mass matrix is also enhanced by the discrete symmetry, so that Eq. (65) is modified to be

$$m_D \sim \begin{pmatrix} \lambda^{L_1+\mathcal{N}_1} & \lambda^{L_1+\mathcal{N}_2} & \lambda^{L_1+\mathcal{N}_3} \\ \lambda^{L_2+\mathcal{N}_1} & \lambda^{L_2+\mathcal{N}_2} & \lambda^{L_2+\mathcal{N}_3} \\ \lambda^{L_3+\mathcal{N}_1} & \lambda^{L_3+\mathcal{N}_2} & \lambda^{L_3+\mathcal{N}_3-2} \end{pmatrix} v_2 , \quad (72)$$

The light neutrino mass matrix in Eq. (68) is modified so that only the 3-3 entry is enhanced,

$$m_\nu \sim \begin{pmatrix} \lambda^{2L_1} & \lambda^{L_1+L_2} & \lambda^{L_1+L_3} \\ \lambda^{L_1+L_2} & \lambda^{2L_2} & \lambda^{L_2+L_3} \\ \lambda^{L_1+L_3} & \lambda^{L_2+L_3} & \lambda^{2L_3-2} \end{pmatrix} \frac{v_2^2}{\Lambda_L} . \quad (73)$$

The charged lepton mass matrix, Eq. (64), and hence a large mixing angle is needed to diagonalize the [2-3] block. So the large mixing observed in the atmospheric neutrino experiments is accounted for, while the hierarchy of order λ^2 in the second and third generation neutrino masses is obtained. Generalizing to a discrete symmetry Z_m rather than Z_2 can preserve the large neutrino mixing while enhance the heaviest neutrino mass by a factor λ^{-m} .

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